

# A COUNTING PROCESS APPROACH TO REPLACEMENT MODELS

by

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## Abstract

In this paper we present a common set-up for a large class of replacement models for stochastically failing systems. The set-up is based on the theory of counting processes. The replacement rule which minimizes "the total expected discounted cost" is determined.

REPLACEMENT; COUNTING PROCESSES; MARTINGALES; MINIMAL REPAIR; SHOCK MODELS; MINIMUM EXPECTED DISCOUNTED COST.

## 1. Introduction

A large number of publications concerning replacement theory have appeared in recent years. Many of these papers deal with the problem of optimal replacement when there is information available about the underlying condition of the system. A stochastic process is usually assumed to describe this information. Some examples of works in this direction are Taylor (1975), Feldman (1976), Abdel-Hameed and Shimi (1978), Bergman (1978), Zuckerman (1978a,b, 1979), Nummelin (1980), Yamada (1980) and Aven (1983).

In this paper, Section 3, we present a common set-up for a large class of replacement models where information about the condition of the system is built-in. The set-up is based on the theory of counting processes.

In order to see how counting processes appear in replacement models, we shall look at two examples. In the model of Bergman (1978) a counting process  $\{N(t)\}$  is generated by defining  $N(t)$  as 1 or 0 according to whether the system has failed or not in  $[0, t]$ ; in the model of Aven (1983) a counting process  $\{N(t)\}$  is generated by defining  $N(t)$  as the number of failures in  $[0, t]$  when minimal repairs are performed at failures. In both examples it is understood when defining  $N(t)$  that no replacement is performed in  $[0, t]$  (at a replacement the system is replaced by a new and identical system).

The optimality criterion we consider is "the total expected discounted cost". In Section 4 we show that this criterion can be written in a form which is basically the same as the one analyzed by Aven (1982). Having this form a number of results obtained by Aven (1982) follows.

It is possible to transform the set-up and results in this paper to for example the optimality criterion "long run (expected) average cost per unit time" - Aven's (1982) analysis also covers this criterion.

The cost-structure we consider is very simple, cf. e.g. the cost-structure of Aven (1983). It is, however, not difficult to make generalizations here.

The set-up in this paper can also be extended in other directions, for example by allowing  $n > 1$  failure modes instead of one (we then have a multivariate counting process).

The set-up in Section 3 is undoubtedly difficult to read - the reader is therefore recommended to go through Section 3 and the special cases given in Section 5 in parallel. The reader is also recommended to study in detail some of the papers mentioned in Section 5.

Before we present the set-up we give a summary of notation and basic concepts.

## 2. Notation and basic concepts

The reference for this section is Brémaud (1981). Let  $(\Omega, \Sigma, P)$  be a probability space. Let  $(X_i, i \in I)$  be a family of random variables on  $(\Omega, \Sigma)$ . We denote by  $\sigma(X_i, i \in I)$  the smallest  $\sigma$ -field of  $\Sigma$  which makes  $X_i$  measurable for all  $i$ .

Let  $(\mathcal{C}_i, i \in I)$  be a family of sub- $\sigma$ -fields of  $\Sigma$ . We denote by  $\bigvee_{i \in I} \mathcal{C}_i$  the smallest  $\sigma$ -field of  $\Sigma$  which contains  $\bigcup_{i \in I} \mathcal{C}_i$ .

Let  $X$  be a random variable on  $(\Omega, \Sigma)$ . The notation  $I(X \in A)$  is used for the indicator variable for the set  $A$ . The symbol  $\wedge$  is used to denote minimum.

Let  $\{F_t, t \in [0, \infty)\}$  be a non-decreasing family of sub- $\sigma$ -fields of  $\Sigma$ . We say that  $\{F_t\}$  is right-continuous iff  $F_t = \bigcap_{s > t} F_s$  for all  $t$ . A stochastic process  $\{X(t), t \in [0, \infty)\}$  is adapted to  $\{F_t\}$  iff  $X(t)$  is  $F_t$ -measurable for all  $t$ . An  $F_t$ -stopping time is a  $[0, \infty]$ -valued random variable such that  $\{T \leq t\} \in F_t$  for all  $t$ . The  $\sigma$ -field  $F_T$  is defined by

$$F_T = \left\{ A \in \bigvee_{t > 0} F_t; A \cap \{T \leq t\} \in F_t, t > 0 \right\}.$$

We next introduce three classes of processes: progressively measurable processes, martingales and counting processes.

A  $[0, \infty)$ -valued process  $\{X(t), t \in [0, \infty)\}$  is said to be  $F_t$ -progressively measurable iff for all  $s > 0$  the mapping  $(t, \omega) \rightarrow X(t, \omega)$  from  $[0, s] \times \Omega$  to  $[0, \infty)$  is  $B[0, s] \times F_s$ -measurable ( $B[0, s]$  denotes the  $\sigma$ -field of Borel-sets on  $[0, s]$ ).

A real-valued process  $\{X(t), t \in [0, \infty)\}$  is called an  $F_t$ -martingale iff

- (i)  $\{X(t)\}$  is adapted to  $\{F_t\}$ ,
- (ii)  $E|X(t)| < \infty, t \geq 0$ ,
- (iii)  $E[X(t) | F_s] = X(s)$  almost surely (a.s.),  $0 \leq s \leq t$ .

If  $\{N(t), t \in [0, \infty)\}$  is a process such that the paths of  $\{N(t)\}$  are non-decreasing, right-continuous,  $\{0, 1, 2, \dots, \infty\}$ -valued, zero at time zero, and with jumps of size 1 only, then  $\{N(t)\}$  is called a counting process.

Let  $\{A(t), t \in [0, \infty)\}$  be a process such that the paths are non-decreasing, right-continuous,  $[0, \infty]$ -valued and zero at time zero, and  $EA(t) < \infty$  for all  $t$ . For such a process we shall tacitly assume that  $A(t) < \infty$  for all paths.

### 3. The set-up

Let  $(\Omega, \Sigma, P)$  be a complete probability space and let  $\{\tilde{F}_t, t \in [0, \infty)\}$  be a non-decreasing family of sub- $\sigma$ -fields of  $\Sigma$  such that  $\tilde{F}_0$  includes all null-sets. The  $\sigma$ -field  $\tilde{F}_t$  represents the total information about the system at  $t$  when no replacement is performed in  $[0, t]$ .

Let  $\{F_t, t \in [0, \infty)\}$  be a non-decreasing family of sub- $\sigma$ -fields of  $\Sigma$  such that  $F_t \subset \tilde{F}_t, t > 0$ . We assume that  $\{F_t\}$  is right-continuous and that  $F_0$  includes all null-sets.

Denote by  $\mathcal{T}'$  the class of all  $F_t$ -stopping times. We shall consider the class of replacement times consisting of all  $\tilde{F}_t$ -stopping times of the form  $T \wedge S$ , where  $T \in \mathcal{T}'$  and  $S$  is a fixed  $\tilde{F}_t$ -stopping time such that  $0 < S \leq U_\infty \stackrel{\text{def}}{=} \inf\{t > 0, N(t) = \infty\}$  ( $\inf \emptyset = \infty$ ).

We assume that a replacement costs  $c$  ( $c > 0$ ) and takes negligible time. Let  $\{N(t), t \in [0, \infty)\}$  be a counting process adapted to  $\{\tilde{F}_t\}$  -  $N(t)$  represents the number of system failures in  $[0, t]$  when no replacement is performed in  $[0, t]$ . Let

$$U_n = \inf\{t > 0; N(t) > n\} \quad (\inf \emptyset = \infty), \quad n = 1, 2, \dots$$

Clearly  $U_n$  represents the time point of the  $n$ -th failure. We assume that each failure costs  $K$  ( $K > 0$ ). Let  $\{\lambda(t), t \in [0, \infty)\}$  be a non-negative  $\tilde{F}_t$ -progressively measurable process such that a.s.

$\lambda(t)$  is non-decreasing in  $t$  for  $t < S$ .

We now formulate the main assumption of this set-up:

$$(3.1) \quad \{N(t \wedge U_n) - \int_0^{t \wedge U_n} \lambda(u)g(u)du\} \text{ is an } \tilde{F}_t\text{-martingale for all } n < \infty,$$

where  $g(t) = I(t \leq U_\infty)$  or  $g(t) = I(t \leq S)$ .

Note that if  $EN(t) < \infty$  for all  $t$ , then (3.1) is equivalent to

$$\{N(t) - \int_0^t \lambda(u)g(u)du\} \text{ is an } \tilde{F}_t\text{-martingale}$$

(this result is proved by using Theorem 6, p.10 and Theorem 8 and 9, pp.27-28 of Brémaud (1981)).

Note also that if  $\lambda(\cdot)$  has right-hand limits and is bounded by an integrable random variable then

$$\lim_{h_n \downarrow 0} E[(N(t+h_n) - N(t)) | \tilde{F}_t] / h_n = \lambda(t^+)g(t^+) \text{ a.s.,}$$

cf. Brémaud (1981), p.28. The process  $\{\lambda(t)g(t)\}$  is called the  $\tilde{F}_t$ -intensity of  $\{N(t)\}$ .

Let  $\alpha$  be a positive discount factor (a cost  $d$  at time  $t$  has value  $de^{-\alpha t}$  at time 0) and let for each  $T \in \mathcal{T}'$ ,

$$(3.2) \quad B^T = \frac{E\left[\int_0^{T \wedge S} Ke^{-\alpha t} dN(t) + ce^{-\alpha(T \wedge S)}\right]}{1 - E e^{-\alpha(T \wedge S)}} \quad \left(\int_0^b \stackrel{\text{def}}{=} \int_{[0, b] \cap [0, \infty)}\right).$$

We interpret  $B^T$  as the total discounted cost in  $[0, \infty)$  when the replacement time  $T \wedge S$  is used, cf. e.g. Aven (1983).

The problem is to find a  $T^* \in \mathcal{T}'$  which minimizes  $B^T$ ,  $T \in \mathcal{T}'$ .

#### 4. Optimal replacement

We begin with the following theorem.

Theorem 1. For any  $T \in T'$ ,

$$(4.1) \quad B^T = \frac{E\left[\int_0^T a(t)h(t)dt + c\right]}{E\left[\int_0^T h(t)dt\right]},$$

where

$$\begin{aligned} \alpha a(t) &= K\lambda(t) - c\alpha \\ h(t) &= \alpha e^{-\alpha t} I(t \leq S). \end{aligned}$$

Proof. Clearly  $1 - Ee^{-\alpha(T \wedge S)} = E \int_0^T h(t)dt$ , hence it suffices to consider the numerator of  $B^T$ . Let  $X(t) = Ke^{-\alpha t} I(t \leq T \wedge S)$ . Then  $\{X(t)\}$  is left-continuous and adapted to  $\{\tilde{F}_t\}$  ( $\{X(t)\}$  is therefore  $\tilde{F}_t$ -predictable). It follows from the results of Brémaud and Jacod (1977), p.369, cf. Brémaud (1981), T9, p.28, that

$$(4.2) \quad E \int_0^\infty X(t)dN(t) = E \int_0^\infty X(t)\lambda(t)g(t)dt$$

(in Brémaud and Jacod's set-up it is assumed that the family of  $\sigma$ -fields is right-continuous; this assumption is not needed for proving (4.2)). It is now easily seen that the numerator of (3.2) equals the numerator of (4.1). The proof of the theorem is completed.

The optimality criterion is now written in a form which is basically the same as the one investigated by Aven (1982) - a number of results follows (it is easily seen by studying Aven's (1982) minimization problem that all the results of Aven (1982) in fact holds for the minimization problem (4.1)). Before we state the results of Aven (1982) we need some definitions.

Let the  $F_t$ -stopping times,  $T_\lambda$ ,  $\lambda \in (-\infty, \infty)$  be defined by

$$T_\lambda = \inf\{t > 0, a(t) > \lambda\}$$

(we see that  $T_\lambda$  minimizes  $E\left[\int_0^T a(t)h(t)dt + c\right] - \lambda E\left[\int_0^T h(t)dt\right] = E\left[\int_0^T (a(t) - \lambda)h(t)dt + c\right]$ ). Furthermore, let

$$B(\lambda) = B^{T_\lambda}.$$

Results from Aven (1982): (cf. Nummelin (1980), Section 3 and Bergman (1980), Section 3 and 4.)

R1. The stopping time  $T_{\lambda^*}$ , where  $\lambda^* = \inf_{T \in \mathcal{T}'} B^T$ , minimizes  $B^T$ ,  $T \in \mathcal{T}'$ . The value  $\lambda^*$  is given as the unique solution of the equation  $\lambda = B(\lambda)$ . Moreover, if  $\lambda > \lambda^*$  then  $\lambda > B(\lambda)$ , if  $\lambda < \lambda^*$  then  $\lambda < B(\lambda)$ ,  $B(\lambda)$  is non-increasing for  $\lambda \leq \lambda^*$ , non-decreasing for  $\lambda > \lambda^*$  and  $B(\lambda)$  is left-continuous.

R2. Choose any  $\lambda_1$  such that  $P\{T_{\lambda_1} > 0\} > 0$ , and set iteratively

$$\lambda_{n+1} = B(\lambda_n), \quad n = 1, 2, \dots$$

Then

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda^*.$$

## 5. Some special cases

5.1. "Replace at failure". Assume that

$$N(t) = I(t > S),$$

where  $S$  represents the lifelength of the system;

$$P[S \leq t | \mathcal{F}_\infty] = \int_0^t f(u) du \stackrel{\text{def}}{=} F(t),$$

where  $F_\infty = \bigvee_{t \geq 0} F_t$  and  $\{f(t)\}$  is a non-negative  $F_t$ -progressively measurable process;

$$\lambda(t) = f(t)/(1-F(t)) \quad \left(\frac{0}{0} \stackrel{\text{def}}{=} 0\right);$$

$$g(t) = I(t \leq S)$$

and

$$\tilde{F}_t = F_\infty \vee \sigma(N(u), u \leq t).$$

We remark that  $\{N(t) - \int_0^t \lambda(u)g(u)du\}$  is an  $\tilde{F}_t$ -martingale (this is easily shown by using Brémaud and Jacod's (1977) proposition on page 373).

This special case represents a counting process version of the models of Bergman (1978) and Nummelin (1979, 1980) - Bergman (1978) and Nummelin (1980) consider the optimality criterion "long run average cost per unit time"; Nummelin (1979) consider "total expected discounted cost". The cost-structure in Nummelin's models are more general than ours.

### 5.2. Minimal repair. Assume that

$$S = \inf\{t > 0, N(t) > n\}, \text{ where } n \in \{0, 1, 2, \dots, \infty\}$$

(this means that the system is replaced before the  $n$ -th failure or at the  $n$ -th failure);

$\tilde{F}_t = F_\infty \vee \sigma(N(u), u \leq t)$  ( $F_\infty = \bigvee_{t > 0} F_t$ );  $E \int_0^t \lambda(u) du < \infty, t > 0$ ;  $\{N(t)\}$  is an  $\tilde{F}_t$ -conditional (doubly stochastic) Poisson process with the intensity  $\lambda(t)$ ; i.e.  $N(t) - N(s)$  is Poisson distributed with parameter  $\int_s^t \lambda(u) du$  given  $\tilde{F}_s, 0 \leq s < t$ ; and  $g(t) = I(t \leq U_\infty)$  (we have  $g(t) \equiv 1$  since  $U_\infty = \infty$ ).

It is easily shown that  $\{N(t) - \int_0^t \lambda(u) du\}$  is an  $\tilde{F}_t$ -martingale (see Brémaud (1981), p.23). Notice that  $EN(t) = E \int_0^t \lambda(u) du < \infty$ .

Now, if  $n = \infty$  our set-up is reduced to the minimal repair/replacement model of Aven (1983) with constant costs - the cost-structure of Aven (1983) is much more general than ours.

If  $n = 1$ , this special case is to be considered as identical to Special case 5.1 (the assumption " $E \int_0^t \lambda(u) du < \infty$ " is not made in Special case 5.1).

### 5.3 Shock models. Assume that shocks occur to the system at random times - each shock causes a random amount of damage and these damages accumulate additively. At a shock the system fails with a given probability. A system failure can occur only at the occurrence of a shock.

Let  $T_n$  be a random variable representing the time point of the  $n$ -th shock,  $0 = T_0 < T_n < T_{n+1} < \infty$ ;  $Y_n$ , a random variable representing the amount of damage caused by the  $n$ -th shock,

$0 < Y_n < \infty$ ; and  $W_n$ , a random variable which equals 1 or 0 according to whether the system fails or not at the  $n$ -th shock.

The sequence  $\{(T_n, (Y_n, W_n))\}_{n=1}^\infty$  is a marked point process (see Brémaud (1981), Chapter VIII). Moreover, let

$\mathcal{B}$  = Borel- $\sigma$ -field on  $[0, \infty)$  and  $\mathcal{D} = (\emptyset, \{0\}, \{1\}, \{0, 1\})$ ;

$N_t(C) = \sum_{n=1}^\infty I((Y_n, W_n) \in C) I(T_n \leq t)$ , where  $C \in \mathcal{B} \times \mathcal{D}$ ; and

$\tilde{F}_t^0 = \sigma(N_u(C), 0 \leq u \leq t, C \in \mathcal{B} \times \mathcal{D})$ .

Notice that

$$N_0(t) \stackrel{\text{def}}{=} N_t([0, \infty) \times \{0, 1\})$$

represents the number of shocks in  $[0, t]$  and

$$N(t) \stackrel{\text{def}}{=} N_t([0, \infty) \times \{1\})$$

represents the number of system failures in  $[0, t]$ .

Define

$$X(t) = \sum_{i=1}^{N_0(t)} Y_i ;$$

$X(t)$  represents the accumulated damage at  $t$ . It can be shown that  $\tilde{F}_t^0 = \sigma(X(u), N(u), u \leq t)$  (cf. Proof of T25, p.305 of Brémaud (1981)).

Now, assume that

$$T_{n+1} - T_n \text{ is independent of } \tilde{F}_{T_n}^0 \vee \sigma(Y_{n+1});$$

$$P\{T_{n+1} - T_n \leq t\} = 1 - e^{-\mu t} ;$$

$$Y_{n+1} \text{ is independent of } \tilde{F}_{T_n}^0 ;$$

$$P\{Y_n \leq y\} = F(y) \quad \text{and}$$

$$(5.1) \quad P[W_{n+1}=1 | \tilde{F}_{T_n}^0 \vee \sigma(T_{n+1}, Y_{n+1})] = 1 - r_{N(T_n)}(X(T_n) + Y_{n+1}) \quad \text{a.s.}$$

where  $r_k(x)$  is a right-continuous function for each  $k$  ( $0 \leq r_k(x) \leq 1$ ).

Note that  $\tilde{F}_{T_n}^0 = \sigma((T_i, (Y_i, W_i)), 1 \leq i \leq n)$  (see T30, p.307 of Brémaud (1981)) and that  $\{N_0(t)\}$  is a Poisson process with intensity  $\mu$ .

Note also that  $N(T_n) = \sum_{i=1}^n W_i$  and  $X(T_n) = \sum_{i=1}^n Y_i$ .

The assumption (5.1) says, roughly speaking, that if the accumulated damage is  $x$  and the number of failures are  $k$  and a shock occurs which causes an amount of damage  $y$ , then the system fails with probability  $1 - r_k(x+y)$ .

Define the process  $\{\tilde{\lambda}(t)\}$  by

$$\tilde{\lambda}(t) = \mu \int_0^\infty (1 - r_{N(t)}(X(t) + y)) dF(y).$$

We see that if  $r_k(x)$  is non-increasing in  $x$  for each  $k$  and non-increasing in  $k$  for each  $x$ , then  $\tilde{\lambda}(t)$  is non-decreasing.

We shall now show by using T7, p.239 of Brémaud (1981) that



$$(5.2) \quad \{N(t) - \int_0^t \tilde{\lambda}(u) du\} \text{ is an } \tilde{F}_t^0\text{-martingale.}$$

Let  $g^{n+1}(u, C)$  ( $C \in \mathcal{B} \times \mathcal{D}$ ) be defined by

$$(5.3) \quad g^{n+1}(u, B \times D) = \mu e^{-\mu u} \int_B \rho_Y^{n+1}(D) dF(Y), \text{ where}$$

$$\rho_Y^{n+1}(D) = I(\{1\} \in D)(1 - r_{N(T_n)}(X(T_n) + Y)) + I(\{0\} \in D)r_{N(T_n)}(X(T_n) + Y)$$

(by Fubini's theorem, T13, p.272 of Brémaud (1981)), we have that

$$g^{n+1}(u, C) = \mu e^{-\mu u} \iint I((x, y) \in C) \rho_Y^{n+1}(dx) dF(y), \quad C \in \mathcal{B} \times \mathcal{D}.$$

Then

$$P[T_{n+1} - T_n \in A, Y_{n+1} \in B, W_{n+1} \in D | \tilde{F}_{T_n}^0] = \int_A g^{n+1}(u, B \times D) du \quad \text{a.s.}$$

This assertion holds since (a.s.)

$$\begin{aligned} & E[I(T_{n+1} - T_n \in A) I(Y_{n+1} \in B) I(W_{n+1} \in D) | \tilde{F}_{T_n}^0] = \\ &= E[I(T_{n+1} - T_n \in A) I(Y_{n+1} \in B) E\{I(W_{n+1} \in D) | \tilde{F}_{T_n}^0 \vee \sigma(T_{n+1}, Y_{n+1})\} | \tilde{F}_{T_n}^0] \\ &= E[I(T_{n+1} - T_n \in A) I(Y_{n+1} \in B) \rho_{Y_{n+1}}^{n+1}(D) | \tilde{F}_{T_n}^0] \\ &= E[E\{I(T_{n+1} - T_n \in A) | \tilde{F}_{T_n}^0 \vee \sigma(Y_{n+1})\} I(Y_{n+1} \in B) \rho_{Y_{n+1}}^{n+1}(D) | \tilde{F}_{T_n}^0] \\ &= EI(T_{n+1} - T_n \in A) E[I(Y_{n+1} \in B) \rho_{Y_{n+1}}^{n+1}(D) | \tilde{F}_{T_n}^0] \\ &= \int_A \mu e^{-\mu u} du \int_B \rho_Y^{n+1}(D) dF(Y) = \int_A g^{n+1}(u, B \times D) du \end{aligned}$$

(a formal proof of the last but one equality can be given by using the result stated in Aven's (1983) appendix).

Let  $C_1 = [0, \infty) \times \{1\}$  and let

$$\lambda_t(C_1) = \sum_{n=0}^{\infty} \frac{g^{n+1}(t - T_n, C_1)}{1 - \int_0^{t - T_n} g^{n+1}(u, [0, \infty) \times \{0, 1\}) du} I(T_n \leq t < T_{n+1}).$$

By using (5.3) we find that

$$\lambda_t(C_1) = \sum_{n=0}^{\infty} \frac{\mu e^{-\mu(t - T_n)} \int_0^{\infty} (1 - r_{N(T_n)}(X(T_n) + Y)) dF(Y)}{1 - \int_0^{t - T_n} \mu e^{-\mu u} du} I(T_n \leq t < T_{n+1})$$

$$= \mu \int_0^{\infty} (1-r_{N(t)}(X(t)+y)) dF(y) = \tilde{\lambda}(t).$$

Now, from Theorem T7, p.239 of Brémaud (1981) we can conclude that

$$\{N(t \wedge T_n) - \int_0^{t \wedge T_n} \tilde{\lambda}(u) du\} \text{ is an } \tilde{F}_t^0\text{-martingale}$$

(remember that  $N(t) = N_t(C_1)$ ). But this implies, since  $EN(t) < \infty$  (we have  $EN(t) < EN_0(t) = \mu t < \infty$ ), that  $\{N(t) - \int_0^t \tilde{\lambda}(u) du\}$  is an  $\tilde{F}_t^0$ -martingale. The proof of (5.2) is completed.

Put  $\tilde{F}_t = \tilde{F}_t^0 \vee \sigma(\text{all null sets})$ . Then  $\{N(t) - \int_0^t \tilde{\lambda}(u) du\}$  is an  $\tilde{F}_t$ -martingale.

Suppose  $r_k(x)$  is non-increasing in  $x$  for each  $k$  and non-increasing in  $k$  for each  $x$ ,  $g(t) = I(t \leq U_{\infty})$  (we have  $g(t) \equiv 1$  since  $U_{\infty} = \infty$ ),  $\lambda(t) = \tilde{\lambda}(t)$ ,  $F_t = \tilde{F}_t$  ( $\{F_t\}$  is right-continuous by Corollary A.2.1, Appendix 2 of Gill (1980)) and  $S = \infty$ . We then have given a special case of the set-up in Section 3. Be aware of the fact that this special case models a system which is repaired at failures (a repair takes negligible time).

Now, suppose  $r_k(x) = r(x)$  for all  $k$ , where  $r(x)$  is a non-increasing function in  $x$ ;  $g(t) = I(t \leq U_{\infty}) \equiv 1$ ;  $\lambda(t) = \tilde{\lambda}(t)$  and  $F_t = \sigma(X(u), u \leq t) \vee \sigma(\text{all null sets})$ . Then if  $S = \infty$ , we have given a special case which represents a kind of minimal repair model; cf. the model of Aven (1983). If  $S = U_1 = \inf\{t > 0, N(t) > 1\}$  we have given a special case which represents a counting process version of the model of Taylor (1975) (Taylor (1975) consider the optimality criterion "long run average cost per unit time"), cf. also Zuckerman (1979) and Yamada (1980). It should be mentioned that if  $S = U_1$  then an "equivalent" model can be generated by letting  $1-r_k(x) = I(k=0)(1-r(x))$ ;  $g(t) = I(t \leq S)$ ,  $\lambda(t) = \mu \int_0^{\infty} (1-r(X(t)+y)) dF(y)$  and  $F_t = \sigma(X(u), u \leq t) \vee \sigma(\text{all null sets})$ .

We emphasise that none of the special cases given here are included in Special case (5.1) or (5.2) - the assertion made by Bergman (1978) that his model includes the model of Taylor (1975) can not be correct.

As far as we know no shock model has been viewed as a marked point process  $\{T_n, (Y_n, W_n)\}$  before; the "incorporation of repairs" in shock models seems also to be new.

Finally we remark that the special cases presented here can be extended in many directions, for example by letting  $\{X(t)\}$  be a more general jump process.

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